Problem 1.26

Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$.

(b) $T_b = \sin x \sin y \sin z$.

(c) $T_c = e^{-5x} \sin 4y \cos 3z$.

(d) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

Solution

Evaluate the Laplacian of a scalar function explicitly.

$$\begin{aligned}
\nabla^2 T &= (\nabla \cdot \nabla)T \\
&= \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right)\right] T \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right] T \\
&= \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right) T \\
&= \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\right) T \\
&= \sum_{i=1}^{3} \frac{\partial^2 T}{\partial x_i^2} \\
&= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}
\end{aligned}$$

Part (a)

Use this formula to find the Laplacian of T_a.

$$\begin{aligned}
\nabla^2 T_a &= \frac{\partial^2}{\partial x^2}(x^2 + 2xy + 3z + 4) + \frac{\partial^2}{\partial y^2}(x^2 + 2xy + 3z + 4) + \frac{\partial^2}{\partial z^2}(x^2 + 2xy + 3z + 4) \\
&= \frac{\partial}{\partial x}(2x + 2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(3) \\
&= (2) + (0) + (0) \\
&= 2
\end{aligned}$$

Part (b)

Use this formula to find the Laplacian of T_b.

$$\begin{aligned}
\nabla^2 T_b &= \frac{\partial^2}{\partial x^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial y^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial z^2}(\sin x \sin y \sin z) \\
&= \frac{\partial}{\partial x}[(\cos x)\sin y \sin z] + \frac{\partial}{\partial y}[\sin x(\cos y)\sin z] + \frac{\partial}{\partial z}[\sin x \sin y(\cos z)] \\
&= [(-\sin x)\sin y \sin z] + [\sin x(-\sin y)\sin z)] + [\sin x \sin y(-\sin z)] \\
&= -3\sin x \sin y \sin z
\end{aligned}$$

Part (c)

Use this formula to find the Laplacian of T_c.

$$\begin{aligned}
\nabla^2 T_c &= \frac{\partial^2}{\partial x^2}(e^{-5x}\sin 4y \cos 3z) + \frac{\partial^2}{\partial y^2}(e^{-5x}\sin 4y \cos 3z) + \frac{\partial^2}{\partial z^2}(e^{-5x}\sin 4y \cos 3z) \\
&= \frac{\partial}{\partial x}[(-5e^{-5x})\sin 4y \cos 3z] + \frac{\partial}{\partial y}[e^{-5x}(4\cos 4y)\cos 3z] + \frac{\partial}{\partial z}[e^{-5x}\sin 4y(-3\sin 3z)] \\
&= [(25e^{-5x})\sin 4y \cos 3z] + [e^{-5x}(-16\sin 4y)\cos 3z] + [e^{-5x}\sin 4y(-9\cos 3z)] \\
&= (25 - 16 - 9)e^{-5x}\sin 4y \cos 3z \\
&= 0
\end{aligned}$$

Part (d)

Evaluate the Laplacian of a vector function explicitly.

$$\begin{aligned}
\nabla^2 \mathbf{v} &= \left(\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\right)\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial^2 v_j}{\partial x_i^2} \\
&= \sum_{i=1}^{3}\left(\boldsymbol{\delta}_1 \frac{\partial^2 v_1}{\partial x_i^2} + \boldsymbol{\delta}_2 \frac{\partial^2 v_2}{\partial x_i^2} + \boldsymbol{\delta}_3 \frac{\partial^2 v_3}{\partial x_i^2}\right) \\
&= \boldsymbol{\delta}_1 \frac{\partial^2 v_1}{\partial x_1^2} + \boldsymbol{\delta}_2 \frac{\partial^2 v_2}{\partial x_1^2} + \boldsymbol{\delta}_3 \frac{\partial^2 v_3}{\partial x_1^2} \\
&\quad + \boldsymbol{\delta}_1 \frac{\partial^2 v_1}{\partial x_2^2} + \boldsymbol{\delta}_2 \frac{\partial^2 v_2}{\partial x_2^2} + \boldsymbol{\delta}_3 \frac{\partial^2 v_3}{\partial x_2^2} \\
&\quad + \boldsymbol{\delta}_1 \frac{\partial^2 v_1}{\partial x_3^2} + \boldsymbol{\delta}_2 \frac{\partial^2 v_2}{\partial x_3^2} + \boldsymbol{\delta}_3 \frac{\partial^2 v_3}{\partial x_3^2}
\end{aligned}$$

Write it in terms of x, y, and z.

$$\nabla^2\mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

For the special case that $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$, $v_x = x^2$ and $v_y = 3xz^2$ and $v_z = -2xz$.

$$\begin{aligned}
\nabla^2\mathbf{v} &= \hat{\mathbf{x}}\left[\frac{\partial^2}{\partial x^2}(x^2) + \frac{\partial^2}{\partial y^2}(x^2) + \frac{\partial^2}{\partial z^2}(x^2)\right] + \hat{\mathbf{y}}\left[\frac{\partial^2}{\partial x^2}(3xz^2) + \frac{\partial^2}{\partial y^2}(3xz^2) + \frac{\partial^2}{\partial z^2}(3xz^2)\right] \\
&\quad + \hat{\mathbf{z}}\left[\frac{\partial^2}{\partial x^2}(-2xz) + \frac{\partial^2}{\partial y^2}(-2xz) + \frac{\partial^2}{\partial z^2}(-2xz)\right] \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(3z^2) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(6xz)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(-2z) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-2x)\right] \\
&= \hat{\mathbf{x}}\left[(2) + (0) + (0)\right] + \hat{\mathbf{y}}\left[(0) + (0) + (6x)\right] + \hat{\mathbf{z}}\left[(0) + (0) + (0)\right] \\
&= 2\hat{\mathbf{x}} + 6x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$